

ASYMPTOTIC FREEDOM OF GENERAL RELATIVITY AND ITS POSSIBLE CONSEQUENCES

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The formation of singularities in certain situations, such as the collapse of massive stars, is one of the unresolved issues in classical general relativity. Although no complete theory of quantum gravity exists, it is often suggested that quantum gravity effects may prevent the formation of these singularities. In this article we will present arguments that a quantized theory of gravity might exhibit asymptotic freedom. Considering the similarities between non-Abelian gauge theories and general relativity it is conjectured that a quantized theory of gravity may have a coupling strength which decreases with increasing energy scale. Such a scale dependent coupling strength, could provide a concrete mechanism for preventing the formation of singularities.

1. Introduction

During the end stages in the evolution of certain supermassive stars general relativity indicates that all the material of the star will collapse into a singularity. This is one of the difficulties with classical general relativity, and it is often suggested that quantum gravity effects will somehow prevent the formation of true singularities. Rhoades and Ruffini¹ have shown that even if the material of the star “stiffens” to the point where the speed of sound in the material becomes equal to the speed of light, the formation of a singularity can not be avoided if the star’s final mass is $\geq 3.2M_{\odot} \approx 6.4 \times 10^{30}$ kg. The Hawking-Penrose theorems² show that the formation of such singularities is a generic feature of classical general relativity. A rough argument for why a singularity inevitably forms for certain collapsing stars can be given as follows : For a star in which the thermonuclear fire has gone out, the gravitational attraction can be counterbalanced by the quantum mechanical Pauli exclusion pressure. To get an estimate of how the quantum mechanical pressure

balances gravity one can use energy considerations³ with the total energy of the star taken as the sum of the energies of all the particles and the gravitational binding energy. In the relativistic regime the average energy of each particle is on the order of cp_F where $p_F = (3\pi^2\hbar^3 N/V)^{1/3}$ is the Fermi momentum, and N is the total number of particles contained in the volume V . The total energy coming from this source is $E_F = Ncp_F$ or

$$E_F = N \left(\frac{3\pi^2 c^3 \hbar^3 N}{V} \right)^{1/3} \quad (1)$$

The gravitational binding energy is of the order, $-GM^2/R$, where M is the total mass and R is the radius of the star. Combining the gravitational binding energy and the energy of Eq. (1) gives an estimate for the total energy of the system (ignoring the rest mass)

$$E_{total} = \left(\frac{9\pi c^3 N^4 \hbar^3}{4} \right)^{1/3} \frac{1}{R} - \frac{GM^2}{R} \quad (2)$$

In the nonrelativistic case the first term goes as $1/R^2$ ³ and there is a radius where stable equilibrium is achieved. For the relativistic case given in Eq. (2), the quantum mechanical exclusion pressure becomes too “soft” so that no stable equilibrium exists and the star collapses. More rigorous work bears out the conclusion of this rough estimate. In addition to Ref.¹, Buchdahl⁴⁻⁵ has shown that a star of mass M with a radius $R = 9M/4$ or smaller, can not reach static equilibrium. These works indicate that the formation of a singularity can not be prevented by the mechanical forces that the material of the star could exert.

Another example of how general relativity results in particles being inevitably forced to the central singularity of a gravitating point mass can be seen by considering a test particle moving in the Schwarzschild field of some point mass M . The effective potential per unit mass is⁵

$$V_{eff} = \frac{c^2}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3} \quad (3)$$

The second term is the standard Newtonian gravitational potential per unit mass, and the third term is the usual centripetal barrier. The last term is a general relativistic addition. It has the effect that if r becomes too small there is no stable orbit, and the particle ends up at $r = 0$. This is to be contrasted with Newtonian gravity where as long as $L^2 \neq 0$ the test particle will not be pulled into the singularity.

One option for avoiding these singularities is that gravity must somehow be modified, and it is usually hypothesized that quantum gravity will somehow accomplish this. In particular one would like the strength of the gravitational interaction to decrease at small distance, or large energies. In the following sections we will present

various arguments that point to the possibility that the gravitational interaction does decrease with decreasing distance.

2. Scaling of G by Analogy with Particle Physics

Gauge theories play an important role in modern physics. In the Standard Model ⁶ of particle physics, matter particles interact via gauge interactions of the group $SU(3) \times SU(2) \times U(1)$. General relativity can also be cast in the form of a gauge theory ⁷. One key difference between the gauge theories of particle physics and general relativity is that the former have been successfully quantized, but not the latter.

When the gauge theories of particle physics are quantized certain phenomenon occur. In particular, the coupling strength of the gauge theory becomes energy or scale dependent. For the electromagnetic part of the Standard Model the coupling strength increases with increasing energy for the energies so far probed. This is seen experimentally ⁸ where at low energies $e^2/4\pi = \alpha_{em} \approx 1/137$ while at energies around 100 GeV $\alpha_{em} \approx 1/128$. For the $SU(3)$, strong interaction part of the Standard Model one finds that the coupling strength decreases with increasing energy. This decrease of the coupling strength with increasing energy is called asymptotic freedom ⁹, and its discovery was one of the first successes of QCD, since it gave an explanation for why, at high energies, the quarks inside the nucleon behaved as if they were essentially free (Bjorken scaling) ¹⁰. Thus, quantized gauge theories have coupling strengths which are scale dependent, and non-Abelian gauge theories can exhibit a coupling strength which becomes weaker at short distances or high energies.

If general relativity is viewed as a gauge theory ¹¹ one can speculate that, as in the case of other quantized gauge theories, the coupling strength of a quantum theory of gravity may become scale dependent. Since general relativity shares similarities with non-Abelian gauge theories, it could be conjectured that general relativity may also be asymptotically free. One of the first theoretical questions which any full theory of quantum gravity, such as string theory or loop quantum gravity ¹², should address is whether the gravitational interaction strength scales with energy, and the nature of the scaling.

3. Scaling in effective theories of general relativity

Effective field theory techniques allow one to discuss the quantum corrections to field theories even if the field theories are conventionally non-renormalizable. Recently Donoghue ¹³ applied effective field theory methods to general relativity to calculate the lowest order quantum corrections to the Newtonian potential. For two point mass, M_1 and M_2 separated by a distance r , quantum corrections modify

the Newtonian potential as

$$V(r) = -\frac{GM_1M_2}{r} \left[1 - \frac{127}{30\pi^2} \frac{G\hbar}{c^3r^2} \right] \quad (4)$$

In Ref. ¹³ there is a further correction which goes as $G(M_1 + M_2)/rc^2$. However this is just a post-Newtonian correction from classical general relativity rather than a quantum correction, since it does not contain \hbar . When a correct theory of quantum gravity is found it should yield the same kind of quantum corrections in the regime where the effective field theory calculation is valid (*i.e.* for r significantly larger than the Planck length). From Eq. (4) it can be seen that the quantum effects tend to decrease the strength of the gravitational interaction as r gets smaller. At ordinary distances this effective decrease of the gravitational coupling is currently unmeasurable since the second term in Eq. (4) is extremely small. It is possible to write Eq. (4) in the usual form, $V(r) = -G(r)M_1M_2/r$, with a r dependent G

$$G(r) = G_\infty \left(1 - \frac{127}{30\pi^2} \frac{G_\infty \hbar}{c^3r^2} \right) \quad (5)$$

where $G_\infty \approx 6.67 \times 10^{-11} Nm^2/kg^2$ is the gravitational coupling constant determined as $r \rightarrow \infty$. For distances not too close to the Planck scale, Eq. (5) implies that Newton's constant decreases with decreasing distance. Usually the running of the coupling constant in gauge theories is given in terms of a scaling with energy rather than with distance. In the appropriate units one can replace distances r for energies k via $r \rightarrow 1/k$, in terms of which the running G from Eq. (5) would become $G(k) = G_0 - (AG_0^2)k^2$ ¹⁴ where $A > 0$ is a constant, and $G_0 = G_\infty$ is the gravitational coupling determined as $k \rightarrow 0$.

In terms of the effective potential of Eq. (3) one can replace G by $G(r)$ of Eq. (5) so that the effective potential becomes

$$\tilde{V}_{eff} = \frac{c^2}{2} - \frac{G(r)M}{r} + \frac{L^2}{2r^2} - \frac{G(r)ML^2}{c^2r^3} \quad (6)$$

Now, whereas V_{eff} of Eq. (3) had no stable minimum if r became too small, \tilde{V}_{eff} of Eq. (6) always has a stable minimum at some small r . This is most easily seen in the $L = 0$ case where, by using Eq. (6) to calculate $d\tilde{V}_{eff}/dr = 0$, one finds that the effective potential with the distance dependent G has a minimum at

$$r_{min} = \sqrt{\frac{127G_\infty\hbar}{10\pi^2c^3}} \approx 1.8 \times 10^{-35} \text{ meters} \quad (7)$$

The numerical value for r_{min} shows the weak point of this hypothesized asymptotic freedom of general relativity : this distance is at the Planck distance scale where the effective theory used to calculate $G(r)$ of Eq. (5) is suspect. At this scale

one really needs a full theory of quantum gravity in order to calculate the scale dependence of G with confidence. One can still speculate that this asymptotic freedom, indicated by the effective field theory at low energies, continues to all energy scales for a complete theory of quantum gravity. This is the reverse of speculations in QCD, where the running of α_{QCD} in the high energy regime is often said to imply the increase of the coupling strength at low energies, and therefore confinement. Also it can be pointed out that superficially the effective field theory result is not completely unreasonable. Plugging r_{min} back into Eq. (5) gives a value for the second term of ≈ 0.3 compared to the value of 1 for the first term, so that the first quantum correction is still smaller than the zeroth order classical term.

To make a connection to the Fermi energy argument we need to relate the distance r between two point particles with the radius R of the star. For a star of radius R composed of N particles, the average distance between any two of the particles will be roughly $r = R/(N)^{1/3}$. With this, Eq. (5) can be written as

$$G(R) = G_{\infty} \left(1 - \frac{127}{30\pi^2} \frac{G_{\infty} \hbar N^{2/3}}{c^3 R^2} \right) \quad (8)$$

Replacing G of Eq. (2) with the scale dependent $G(R)$ of Eq. (8) and calculating dE_{total}/dR now gives

$$\frac{dE_{total}}{dR} = \frac{1}{R^2} \left(G_{\infty} M^2 - \left[\frac{9\pi c^3 \hbar^3 N^4}{4} \right]^{1/3} \right) - \frac{127 G_{\infty}^2 \hbar N^{2/3} M^2}{10\pi^2 c^3 R^4} \quad (9)$$

The last term, which arises from the quantum corrections of the effective gravitational field theory, ensures that it is always possible to find some R so that $dE/dR = 0$. This implies that a stable balance between gravity and the quantum mechanical pressure can be achieved due to the weakening of the gravitational interaction. Aside from the heuristic nature of this argument (a more serious calculation would use the Oppenheimer-Volkoff equation with the scale dependent $G(R)$ of Eq. (8)) it is found that the radius R , for which Eq. (9) gives an equilibrium, is again outside the regime where the effective field theory calculation can be trusted. For a star of mass $M = 5M_{\odot} \approx 1.0 \times 10^{31} \text{ kg}$ with $N = M/m_n = 5.97 \times 10^{57}$, it is found that Eq. (9) gives an equilibrium radius of $R = 2.70 \times 10^{-15} \text{ m}$. This implies an average spacing between the particles of $r = R/N^{1/3} = 1.49 \times 10^{-34} \text{ m}$, which is only one order of magnitude above the Planck scale of 10^{-35} m . Again, one can hypothesize that the weakening of the gravitational coupling, G , implied by the low energy effective theory will continue at higher energy scales.

This hypothesized asymptotic freedom for general relativity would not prevent the formation of a black hole, since in the example given above the horizon forms at a distance around 15 km. The scaling of G only replaces the singularity at the center of the black hole with an extremely dense, but non-singular mass.

4. Scaling in Kaluza-Klein Theories

If the gravitational interaction is eventually unified with the Standard Model interactions, then the scaling of the various coupling strengths may be interrelated. This is similar to grand unified theories such as $SU(5)$ ¹⁵, where the scaling of the various non-gravitational couplings are related. Kaluza-Klein theories offer a simple and direct example of how the scaling of the gravitational and non-gravitational coupling strengths may be related. In the original Kaluza-Klein theory¹⁶ a relationship exists between the electric coupling e and Newton's constant

$$G = \frac{r_5^2 e^2}{16\pi} = \frac{r_5^2 \alpha_{em}}{4} \quad (10)$$

where r_5 was the radius of the curled up fifth dimension. To get non-Abelian gauge fields it is necessary to have more than one compactified dimension. In Ref.¹⁷ a relationship similar to Eq. (10) is given except with the electromagnetic coupling e is replaced by the non-Abelian coupling g , and r_5 replaced by some rms circumference of the curled up dimensions. The key point about Eq. (10) or its non-Abelian version, is that Newton's constant is proportional to the square of some non-gravitational coupling constant. Thus G should scale with distance or energy in the same manner as g^2 . For non-Abelian theories the coupling strength, $\alpha = g^2/4\pi$, usually decreases in strength with decreasing distance scale in a logarithmic way (*i.e.* $\alpha(r) = \alpha_o[1 + c\alpha_o \ln(r/r_o)]^{-1}$ where c is some positive constant which depends on the non-Abelian gauge group, and r_o is a reference distance at which the coupling is measured) so that $G(r)$ should also decrease logarithmically. The running of G here is different than in the previous section. First, as noted in Ref.¹⁴ the running of G implied by the effective field theory treatment goes as a power of energy or inverse power of distance, whereas in the present example, the running is logarithmic. Secondly, in the effective field theory approach the direct quantum corrections of gravity were discussed. Here, the *direct* quantum effects of four dimensional gravity are ignored, but one still finds that G runs if g runs. At energies far from the Planck scale the description of the compactified dimensions in terms of a non-Abelian gauge field theory is reasonable, especially if these Kaluza-Klein fields of the compactified dimensions are to describe real non-Abelian fields. If the effective, non-Abelian coupling, g , exhibits asymptotic freedom (as it should if it is to model the behaviour of non-Abelian fields of the Standard Model) then so will G . As in the previous section, when Planck scale energies and distances are approached, this treatment of the curled up dimensions by an effective non-Abelian gauge theory breaks down, and a complete, non-perturbative method of quantizing this higher dimensional gravitation theory is required. This running of the gravitational coupling in Kaluza-Klein models again opens up the possibility that the formation of singularities in gravitational collapse may be avoided.

One worry about this Kaluza-Klein argument is that if the running of the non-Abelian coupling, g , is experimentally observed then the running of G should also be seen. For example, let g be the QCD coupling. The perturbative running of the QCD coupling is experimentally observed at energy scales greater than about 2 GeV (see Ref. ⁸ p. 82). If the running of G were tied to g then one might think that some experimental signature of this running of G should have been seen. In accelerator experiments, however, one deals with such small quantities of matter, gravitationally speaking, that any kind of running of G would be undetectable. Even inside an apparently high energy environment like the interior of the Sun, where there is a gravitationally significant amount of matter, one has a temperature $\approx 1.6 \times 10^7$ K, which corresponds to an energy scale of about 1.4 keV. This is not in the energy range where the perturbative running of g (and therefore G) would apply.

Situations where a gravitationally significant amount of matter at a high enough energy could exist, occur in situations of gravitational collapse. For example, taking a stellar mass of $M = 5M_{\odot} = 1.0 \times 10^{31}$ kg, so that $N = M/m_n = 5.97 \times 10^{57}$, and taking $R = 1000$ m gives, an average energy per particle of $E_F/N \approx 6.9$ GeV (see Eq. (1)), which is an energy range where the perturbative scaling of g should apply.

5. Conclusions

We have argued that the singularities which occur in general relativity in certain situations, such as gravitational collapse, could possibly be avoided if quantum gravity exhibits a scaling of Newton's constant. In the case of stellar collapse it has been shown ¹ that even if the material of the dead star exerts the maximum possible resistive force, it can not counterbalance the inward push of gravity. Thus the only obvious way to avoid these singularities would be to somehow modify the gravitational interaction, which is essentially the idea behind the common statements that a full quantum theory of gravity would somehow prevent the formation of these singularities. In this article we have presented motivations that a quantum theory of gravity should have a coupling strength which weakens at short distance, or large energy scales, thus allowing an equilibrium to be reached between the quantum mechanical exclusion pressure and the weakened gravitational interaction. First, by analogy with other non-Abelian gauge theories, which when quantized exhibit asymptotic freedom, we argued that a quantum theory of gravity may also exhibit asymptotic freedom. Second, from recent effective field calculations it is found that the gravitational interaction does grow weaker with decreasing distance scale, at least for scales which are not too close to the Planck scale. Finally, if gravity is unified with the other interactions, as in Kaluza-Klein theories, then the running

of the different couplings should be related; if the non-Abelian coupling g exhibits asymptotic freedom then so should G . All of these arguments are only good for energies and distances far from the Planck scale. However, the idea that quantum gravity may exhibit asymptotic freedom provides a concrete mechanism of how the singularities of classical general relativity may be avoided. There are currently theories, such as string theory or loop quantum gravity, which hold out the hope of giving a complete quantum theory of gravity. One of the first questions that could be asked of such a complete theory of quantum gravity would be the nature of the non-perturbative scaling, if any, that it gives for G .

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